Social Networks II

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Learning Goals Today / Overview

- Non-overlapping communities: the Girvan-Newman Algorithm
- Overlapping communities: the Graph Affiliation Model

Reminder: Betweenness

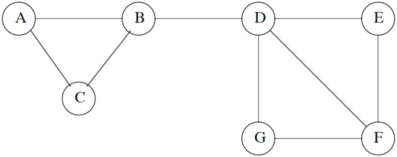
Betweenness

Idea: Identify edges that are least likely to be within community

Definition [Betweenness]
The *betweenness* of an edge (a, b) is

- the number of pairs of nodes (x, y) such that (a, b) makes part of the *shortest path* leading from x to y
- If for (x, y) there are several shortest paths, (a, b) is credited the fraction of shortest paths leading through (a, b) when computing its betweenness

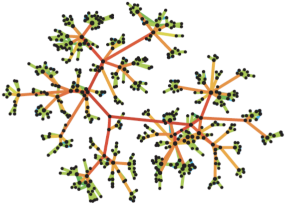
BETWEENNESS: EXAMPLE



Adopted from `mmds.org`

- (B, D) has the greatest betweenness, 12
  - It is on any shortest path between A, B, C and D, E, F, G
- (D, F) has betweenness 4
  - It lies on all shortest paths between A, B, C, D and F

Betweenness



Telephone network:
Links between communities have great betweenness

Adopted from `mmds.org`

Explanation

- High betweenness means that (a, b) is a bottleneck for shortest paths
- If nodes (a, b) lie within community, there are too many options for shortest paths to circumvent (a, b) (so (a, b) gets credited only small fractions)

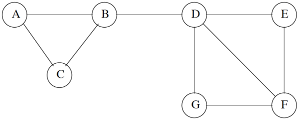
Computing Betweenness
The Girvan-Newman Algorithm

The Girvan-Newman Algorithm

Calculating Betweenness

Algorithmic Principle

- Visit each node X once
- Compute shortest paths from X to any other node Y
- To visit nodes Y from X, perform breadth-first search (BFS)



Social Network; consider BFS from E

Adopted from `mmds.org`

The Girvan-Newman Algorithm

Calculating Betweenness

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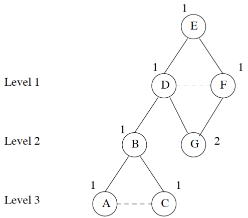
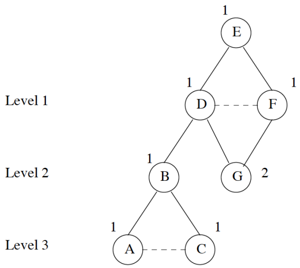


BFS starting from E on social network from slide before

Adopted from `mmds.org`

The Girvan-Newman Algorithm

Calculating Betweenness



BFS starting from *E*

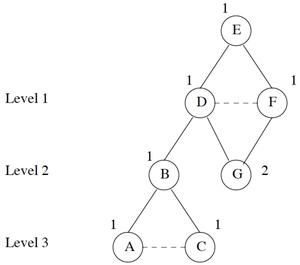
Adopted from `mmds.org`

Intuition / Notation

- Length of shortest path from X to Y: level of BFS starting at X
- Edges within BFS level cannot be part of shortest paths from X
- Edges between different levels are referred to as *DAG (directed acyclic graph) edges*
- DAG edges are on at least one shortest path leaving from X

THE GIRVAN-NEWMAN ALGORITHM

CALCULATING BETWEENNESS



BFS starting from E

Adopted from `mmds.org`

EXAMPLE NOTATION

- Root X = uppermost node, example $X = E$
- Solid edges = DAG edges: e.g. (D, B), (E, F)
- Dashed edges = within level: e.g. (D, F), (A, C)
- For DAG edge (Y, Z) where Y is closer to root X than Z:
  - Y is said to be the *parent*
  - Z is said to be the *child*

The Girvan-Newman Algorithm

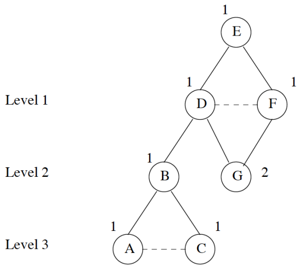
Calculating Betweenness

Two Stages

- *Labeling:* For each node, assign number of shortest paths from root to that node
  - Proceed from root to leaves in BFS order
- *Crediting:* For each edge, compute contribution of shortest paths from root for betweenness of that edge
  - Need to compute credits for nodes as well
  - Proceed from leaves to root, bottom-up

The Girvan-Newman Algorithm

Calculating Betweenness



BFS starting from *E*

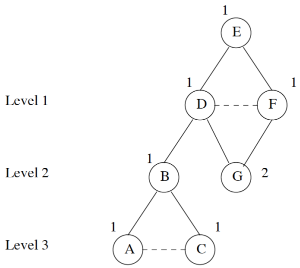
Adopted from `mmds.org`

Labeling Nodes

- Label each node by the number of shortest path to the root
- Start by labeling the root with 1
- Top-down, label each node by the sum of labels of each parents

The Girvan-Newman Algorithm

Calculating Betweenness



BFS starting from *E*: Labeling

Adopted from `mmds.org`

Example Labeling

- Label the *root* E with 1
- *Level 1:* Each D and F have only E as parent; label both with 1
- *Level 2:*
  - B has only D as parent, label with 1
  - G has parents D and F, label with 2
- *Level 3:* Both A, C have only B as parent, so both are labeled with 1

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CREDITING NODES

- Compute fraction of shortest paths from root passing through node
- Credit each *leaf* with 1
  - If several shortest paths run to leaf, fractions add up to 1
- Each *non-leaf node* v gets credit

$$1 + \sum_{e \in \mathcal{D}(v)} c(e) \tag{1}$$

where $\mathcal{D}(v)$ are the DAG edges leaving from v, and $c(e)$ is the credit of an edge e

How to credit edges?

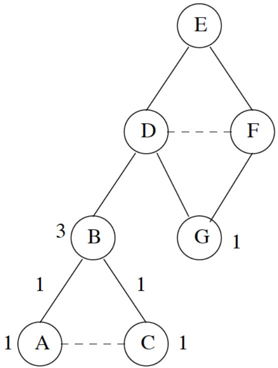
Crediting Edges

- Let $u_j, j = 1, ..., k$ be the parents of w; so (u_j, w) are the DAG edges entering w
- Let $N_j, j = 1, ..., k$ be the number of shortest paths from root running through edges (u_j, w)
- *Recall:* N_j agrees with the *label* of u_j, the number of shortest paths from root to u_j ...
- ... because every shortest path from root to u_j extends to shortest path from root to w
- Let $c(w)$ be the credit of w
- We compute the credit of (u_i, w) as

$$c(u_i, w) := c(w) \times \frac{N_i}{\sum_{j=1}^{k} N_j} \tag{2}$$

The Girvan-Newman Algorithm

Calculating Betweenness



Crediting Nodes and Edges in Level 3 and 2

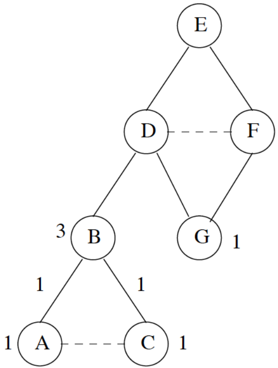
Adopted from `mmds.org`

Example Crediting

- *Level 3 Nodes:* Credit each of nodes A and C with 1
- *Level 2-3 Edges:* Both A and C have only one parent, so full credit 1 is assigned to both (B, A) and (B, C)

THE GIRVAN-NEWMAN ALGORITHM

CALCULATING BETWEENNESS



Crediting Nodes and Edges in Level 3 and 2

Adopted from `mmds.org`

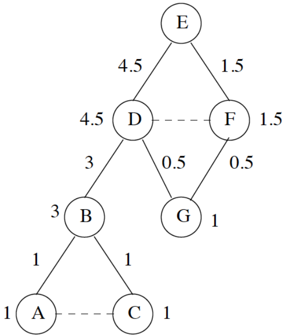
EXAMPLE CREDITING

Level 2 Nodes:

- G is a leaf, so gets credit 1
- B is not a leaf, so gets credit 1 + sum of credits 1 of DAG edges (B, A), (B, C) leaving from it: credit 3 overall
- Intuitively, credit 3 for B refers to all shortest paths from E to A, B, C going through B.

The Girvan-Newman Algorithm

Calculating Betweenness



Crediting Nodes and Edges

Adopted from `mmds.org`

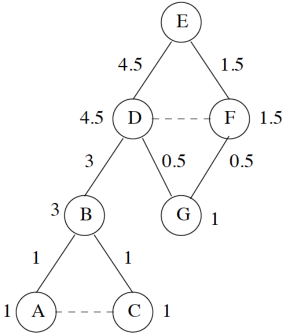
Example Crediting

Level 1-2 Edges:

- B has only one parent, D, so the edge (D, B) gets all of B's credit
- $(D, G), (F, G)$: Both D, F have label (not credit!) 1. So we credit both $(D, G), (F, G)$ with $1/(1 + 1) = 0.5$
- *Example:* If labels of D and F had been 3 and 5, the credit of (D, G) would be $3/(3 + 5) = 3/8$ and that of (F, G) would be $5/8$.

The Girvan-Newman Algorithm

Calculating Betweenness



Crediting Nodes and Edges

Adopted from `mmds.org`

Example Crediting

Level 1 Nodes / Edges:

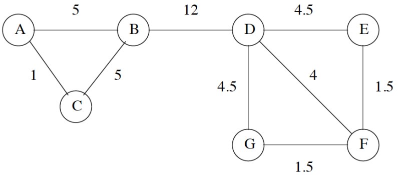
- D gets credit 1 + credits of (D, B), (D, G) = credit 4.5 overall
- F gets credit 1 + credit of (F, G) = credit 1.5 overall
- Edges (E, D), (E, F) receive credits of D, F respectively, because D, F each have only one parent

Summary: Credit on each edge is contribution to betweenness of that edge to shortest paths from E

THE GIRVAN-NEWMAN ALGORITHM
SUMMARY

COMPLETING THE ALGORITHM

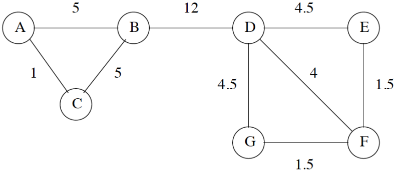
- Repeat the calculation illustrated for E for every other node
- Sum up the contributions for each edge across different roots
- Divide each edge weight by 2: each shortest path is counted twice, with each of its end points as root



Betweenness Scores

Adopted from `mmds.org`

Finding Communities with Betweenness



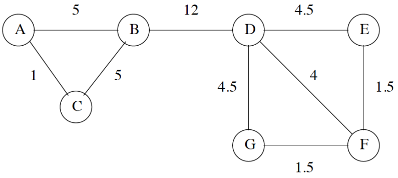
Betweenness Scores

Adopted from `mmds.org`

Computing Communities: Principle

- Remove edges in decreasing order of betweenness
- Stop at reasonably chosen threshold
- Communities are the resulting connected components

Finding Communities with Betweenness



Betweenness Scores

Adopted from `mmds.org`

Computing Communities: Example Threshold 4

- First, remove (B, D): communities $\{A, B, C\}, \{D, E, F, G\}$
- Second, remove $(A, B), (B, C)$: communities $\{A, C\}, \{B\}, \{D, E, F, G\}$
- Third, remove $(D, E), (D, G)$: communities $\{A, C\}, \{B\}, \{D, E, F, G\}$
- Last, remove (D, F): communities $\{A, C\}, \{B\}, \{D\}, \{E, F, G\}$

FINDING COMMUNITIES WITH BETWEENNESS

COMPUTING COMMUNITIES: EXAMPLE THRESHOLD 4

- First, remove (B, D): communities $\{A, B, C\}, \{D, E, F, G\}$
- Second, remove $(A, B), (B, C)$: communities $\{A, C\}, \{B\}, \{D, E, F, G\}$
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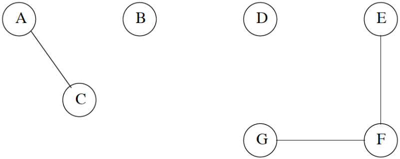
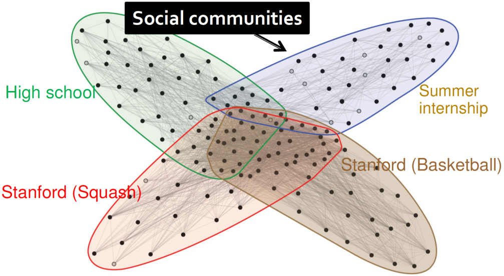


Final Communities

Adopted from `mmds.org`

The Graph Affiliation Model

Overlapping Communities

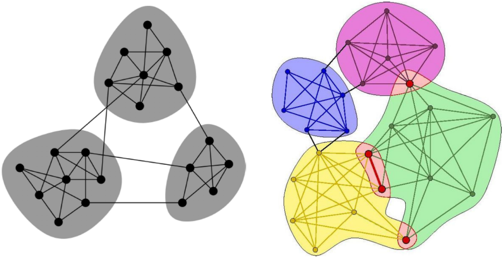


Subgraph from Facebook

Adopted from `mmds.org`

- *Observation:* Communities in social networks can overlap
- Graph partitioning does not help in these cases
- Would like to have a statistical interpretation of network data

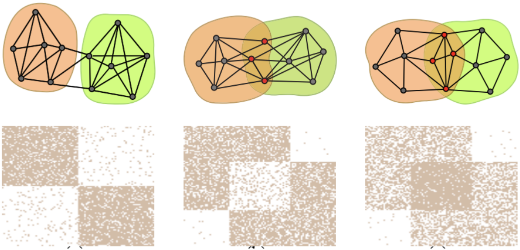
Nonoverlapping versus Overlapping Communities



Left: Nonoverlapping communities
Right: Overlapping communities
Adopted from mmds.org

- Communities may overlap or not
- *Issue:* How to determine communities correctly?

Affiliation Graph Model: Introduction

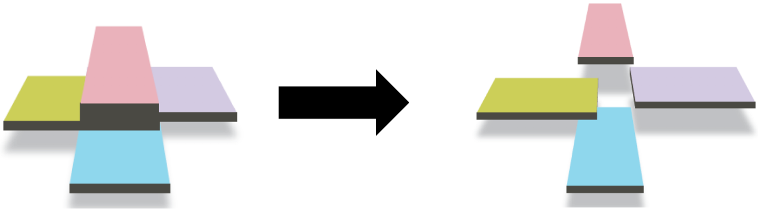


Networks and their adjacency matrices

Adopted from `mmds.org`

- Left: No overlap, adjacency matrix sparse across communities
- Middle: Loose overlap, adjacency matrix less sparse in shared part
- Right: Tight overlap, adjacency matrix dense in shared part

Community Discovery: Goal



Revealing (overlapping) communities

Adopted from `mmds.org`

- *Goal:* Discover communities correctly
- Regardless of whether they overlap or not

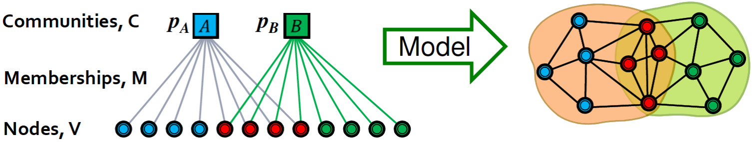
Determine the statistically most likely community structure

Affiliation Graph Model: Introduction

- *Issue:* Statistical control over community structure of a network
- *Idea:* Design *generative probability distribution*
- Given a number of nodes, this generative distribution generates edges
- The generative distribution represents a particular community structure
    - The distribution knows about nodes belonging to communities
    - It generates more edges within communities
    - It generates less edges between communities

Affiliation Graph Model: Introduction

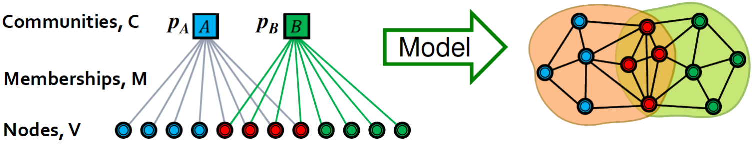
- The generative distribution represents community structures
  - The distribution knows about nodes belonging to communities
  - It generates more edges within communities
  - It generates less edges between communities



Distribution representing a community structure generating network

Adopted from `mmds.org`

Affiliation Graph Model: Introduction



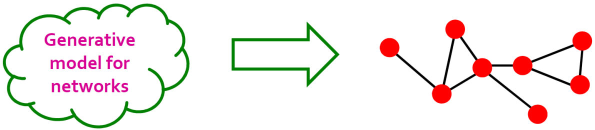
Distribution representing a community structure (left) generating network (right)

Adopted from `mmds.org`

- We can generate networks when knowing community structure
- *But:* We would like to determine the community structure when knowing the network

Isn't that exactly the opposite?

Generative Distributions



We can do this: generating network from distribution...

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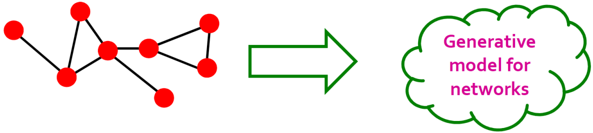

...but we want this: inferring distribution from network

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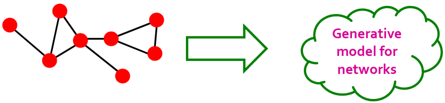
GENERATIVE DISTRIBUTIONS: MAXIMUM LIKELIHOOD INFERENCE



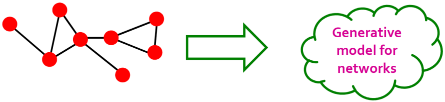
We want to infer distribution from network

Adopted from `mmds.org`

Maximum Likelihood Estimation

- Let Θ be a *parameterized class of probability distributions* that generate networks
  - We identify the different distributions with the different parameterizations
  - ☞ Formally not 100% correct, but doesn't matter here
- Let $\mathbf{P}(N \mid \theta)$ be the probability that distribution $\theta \in \Theta$ generates network N

Generative Distributions: Maximum Likelihood Inference



We want to infer distribution from network

Adopted from `mmds.org`

Maximum Likelihood Estimation

- Let $\mathbf{P}(N \mid \theta)$ be the probability that distribution $\theta \in \Theta$ generates network N
- *Maximum likelihood estimation:* Determine distribution $\hat{\theta}$ that generated N with greatest likelihood:

$$\hat{\theta} := \arg\max_{\theta \in \Theta} \mathbf{P}(N \mid \theta) \tag{3}$$

- This computes most reasonable distribution $\hat{\theta}$ for network N

Affiliation Graph Model: Definition I

- An AGM θ generates a network $N = (V, E)$ by adding edges E to a given set of nodes V
- For $u, v \in V$, edge (u, v) is generated with probability $\mathbf{P}_\theta((u, v))$
- $\mathbf{P}_\theta((u, v))$ depends on the parameters θ
- Recall that θ specifies community structure

So, what exactly is θ supposed to be?

Affiliation Graph Model: Parameters

- $\mathcal{C}$, as a set of *communities*
- $M \in \{0,1\}^{\mathcal{C} \times V}$, specifying *assignment of nodes $v \in V$ to communities* $C \in \mathcal{C}$, where

$$M_{C,v} = \begin{cases} 1 & v \text{ belongs to } C \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

  - M specifies "affiliations" of nodes $v \in V$
  - Note that one can vary $\mathcal{C}$, as a parameter, but not V
- $(p_C)_{C \in \mathcal{C}}$ as probabilities to generate edges (u, v) because $u, v \in C$
- *Summary:* A particular AGM θ corresponds to

$$\theta = (\mathcal{C}, M, (p_C)_{C \in \mathcal{C}}) \tag{5}$$

Affiliation Graph Model: $\mathbf{P}_\theta((u, v))$

Several C **containing both** u, v

- Let $M_u, M_v \subset \mathcal{C}$ be the subsets of communities that contain u and v, respectively
- Existence of communities that contain both u, v means

$$M_u \cap M_v \neq \emptyset$$

- Memberships in different communities have no influence on each other
- That is, we assume *statistical independence*

AFFILIATION GRAPH MODEL: $\mathbf{P}_\theta((u, v))$

Several C **containing both** u, v

- Statistical independence is expressed by

$$\prod_{C \in M_u \cap M_v} (1 - p_C)$$

as probability of *no edge* (u, v) *in any community* $C \in M_u \cap M_v$

- Hence, the probability to generate (u, v) is

$$1 - \prod_{C \in M_u \cap M_v} (1 - p_C) \tag{6}$$

Done? No: What about $M_u \cap M_v = \emptyset$?

Affiliation Graph Model: $\mathbf{P}_\theta((u,v))$

No C **containing both** u, v

- For $M_u \cap M_v = \emptyset$, computing (6) yields (empty product is 1)

$$1 - \prod_{C \in \emptyset} (1 - p_C) = 1 - 1 = 0$$

- No edges across communities makes no sense
- Let $\epsilon > 0$ be small; we generate an edge (u, v) with probability

$$\mathbf{P}_\theta((u,v)) = \epsilon \quad \text{if} \quad M_u \cap M_v = \emptyset$$

Affiliation Graph Model: $\mathbf{P}_\theta((u,v))$

Affiliation Graph Model (AGM)

- An edge (u,v) is generated with probability

$$\mathbf{P}_\theta((u,v)) = \begin{cases} 1 - \prod_{C \in M_u \cap M_v} (1 - p_C) & M_u \cap M_v \neq \emptyset \\ \epsilon & M_u \cap M_v = \emptyset \end{cases} \tag{7}$$

- Edges (u,v) are generated independently from one another
- *Overall:* The probability $\mathbf{P}_\theta(E)$ to generate edges E given AGM θ computes as

$$\mathbf{P}_\theta(E) = \prod_{(u,v) \in E} \mathbf{P}_\theta((u,v)) \times \prod_{(u,v) \notin E} 1 - \mathbf{P}_\theta((u,v)) \tag{8}$$

where $\mathbf{P}_\theta((u,v))$ are computed following (7), with $\theta = (\mathcal{C}, M, p_C)$ determining p_C and M_u, M_v and so on.

Affiliation Graph Model: Overall Probability

Affiliation Graph Model (AGM)

- The probability $\mathbf{P}_\theta(E)$ to generate E given θ is

$$\mathbf{P}_\theta(E) = \prod_{(u,v)\in E} \mathbf{P}_\theta((u,v)) \times \prod_{(u,v)\notin E} 1 - \mathbf{P}_\theta((u,v)) \tag{9}$$

- *Reminder:* For a given network $N = (V, E)$, the *goal* is to determine

$$\hat{\theta} := \arg\max_{\theta\in\Theta} \mathbf{P}_\theta(E)$$

- That is, we need to vary $\theta = (\mathcal{C}, M, p_C)$ until $\mathbf{P}_\theta(E)$ is maximal

How to *systematically vary* $\theta = (\mathcal{C}, M, p_C)$?

Computing the MLE $\hat{\theta}$

Issues

- Search space of combinations of
  - Communities $\mathcal{C}$,
  - Assignments of nodes to communities M, and
  - Probabilities p_C for communities

  tends to be huge
- Concise formulas of (9) for $\mathbf{P}_\theta(E)$ as function of θ too difficult
- Analytical solution for determining $\hat{\theta} := \arg\max_{\theta \in \Theta} \mathbf{P}_\theta(E)$ not available
- Moreover, parameters are both discrete $(\mathcal{C}, M)$ and continuous $((p_C)_{C \in \mathcal{C}})$

Computing the MLE $\hat{\theta}$

Approach

1. Pick initial set of parameters θ_0
2. Vary θ such that $\mathbf{P}_\theta(E)$ iteratively increases
3. Vary $\mathcal{C}$ or M first
   - ☞ Partial derivates of $\mathbf{P}_\theta(E)$ wrt. p_C computable on fixed $\mathcal{C}, M$
4. Determine optimal $(p_C)_{C \in \mathcal{C}}$, e.g. by gradient descent
5. Keep change if $\mathbf{P}_\theta(E)$ has increased, discard otherwise

Computing the MLE $\hat{\theta}$

Iterative variations of $\mathcal{C}, M$

- *Varying M:*
  - Delete node from community, i.e. for $M_{C,v} = 1$, set $M_{C,v} = 0$
  - Add node to community, i.e. for $M_{C,v} = 0$, set $M_{C,v} = 1$
- *Varying $\mathcal{C}$:*
  - Merge two communities
  - Split community
  - Delete community
  - Add new community, with initial random selection of members

Computing the MLE $\hat{\theta}$

Soft Community Membership

- Instead of $M_{C,v} \in \{0,1\}$, allow any real-numbered $M_{C,v} \geq 0$
- For (u,v) to be generated because of $u, v \in C$, let

$$\mathbf{P}_\theta((u,v)) = 1 - e^{-M_{C,u}M_{C,v}} \tag{10}$$

  be the individual probability
- Proceeding exactly as before, we obtain

$$\mathbf{P}_\theta(E) = \prod_{(u,v)\in E} (1 - e^{-\sum_C M_{C,u}M_{C,v}}) \prod_{(u,v)\notin E} e^{-\sum_C M_{C,u}M_{C,v}} \tag{11}$$

Computing the MLE $\hat{\theta}$

Soft Community Membership

- Probability for edges E:

$$\mathbf{P}_\theta(E) = \prod_{(u,v)\in E} (1 - e^{-\sum_C M_{C,u} M_{C,v}}) \prod_{(u,v)\notin E} e^{-\sum_C M_{C,u} M_{C,v}} \tag{12}$$

- On fixed communities, include M in gradient descent (or related) optimization step
- *Advantages:*
  - Only one gradient descent run necessary
  - Less prone to get stuck in unfavorable local optima
- If necessary, add or delete communities, and re-run

General / Further Reading

Literature

- Mining Massive Datasets, Sections 10.2, 10.3, 10.5 `http://infolab.stanford.edu/~ullman/mmds/ch10.pdf`
- Next lecture: “Web Advertisements”: sections 8.1 – 8.4 in *Mining of Massive Datasets*