# Frequent Itemsets

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# Learning Goals Today

- The Market-Basket Model
- Frequent Itemsets: Definition and Applications
- Association Rules
- The A-Priori Algorithm
  - Data Representation
  - Runtime and Space Considerations
  - Monotonicity
  - The Algorithm
- The Algorithm of Park, Chen and Yu (PCY)

*Frequent Itemsets*
*Introduction*

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# Frequent Itemsets: Overview

*Foundations*

- There are *items* available in the market
- There are *baskets*, sets of items having been purchased together
- A *frequent itemset* is a set of items that is found to commonly appear in many baskets
- The *frequent-itemset problem* is to identify frequent itemsets

# Market-Basket Model

*Market-basket model*

- The market-basket model is a *many-many-relationship*
  - One basket holds many items
  - One item appears in several baskets
- Each basket is an itemset, i.e. a set of (one or several) items
- Usually, the number of items in a basket is small compared to number of items overall
- Number of baskets is usually large; too large to fit in main memory
- Data usually is a sequence of baskets

# Frequent Itemsets: Definition

Definition [Frequent Itemset]:

- Let $s > 0$ be a *support threshold*
- Let $I$ be a set of items
- $\text{supp}(I)$, the *support* of $I$, is the number of baskets in which $I$ appears as a subset

An itemset $I$ is referred to as *frequent* if

$$\text{supp}(I) \geq s \tag{1}$$

that is, if the support of $I$ is at least the support threshold

# Frequent Itemsets: Example

*Baskets*

1. {and, dog, bites}
2. {news, claims, a, cat, mated, with, a, dog, and, produced, viable, offspring}
3. {cat, killer, likely, is, a, big, dog}
4. {professional, free, advice, on, dog, training, puppy, training}
5. {cat, and, kitten, training, behavior}
6. {dog, cat, provides, training, in, Oregon}
7. {dog, and, cat, is, a, slang, term, used, by, police, officers, for, a, male-female, relationship}
8. {shop, for, your, show, dog, grooming, and, pet, supplies}

- E.g. supp({dog}) = 7, supp({and}) = 5, supp({dog, and}) = 4
- Let the support threshold $s = 3$
- 5 frequent singletons: {dog},{cat},{a},{and},{training}
- 5 frequent doubletons: {dog, a},{dog, and},{dog, cat},{cat, a},{cat, and}
- 1 frequent triple: {dog, cat, a}

# Frequent Itemsets: Applications

- *Retailers / Supermarkets / Chain stores*
  - *Items:* Products offered
  - *Baskets:* Sets of products purchased by one customer during one shopping run
  - *Frequent Itemsets:* Products purchased together unusually often
    ☞ Beer and diapers
- *Related concepts*
  - *Items:* Words, excluding stop words
  - *Baskets:* News articles, documents
  - *Frequent Itemsets:* Groups of words representing joint concept
- *Plagiarism*
  - *Items:* Documents
  - *Baskets:* Sentences
  - *Frequent Itemsets:* Documents containing unusually many sentences in common

# Association Rules

- Let $j$ be an item and $I$ be an itemset
- An association rule
$$I \rightarrow j$$
expresses that if $I$ is likely to appear in a basket, so is $j$
- In other words, if $I$ shows in basket, one is confident to assume that $j$ does, too

Definition [Confidence]:
The *confidence* of a rule $I \rightarrow j$ is defined as

$$\frac{\text{supp}(I \cup \{j\})}{\text{supp}(I)} \tag{2}$$

that is the fraction of $I$ containing baskets that also contain $j$.

# Association Rules: Confidence

Definition [Confidence]:
The *confidence* of a rule $I \to j$ is defined as

$$\frac{\mathrm{supp}(I \cup \{j\})}{\mathrm{supp}(I)}$$

that is the fraction of $I$ containing baskets that also contain $j$.

*Example from above*

- Confidence of $\{cat, dog\} \to and$ is 3/5
- Confidence of $\{cat\} \to kitten$ is 1/6

# Association Rules: Interest

- Let $n$ be the number of baskets overall
- Confidence for $I \rightarrow j$ can be meaningless if fraction of baskets containing $j$ is large
- Confidence may just reflect that fraction
- So presence of $I$ does not increase confidence to see $j$ as well
- *Interest* is supposed to put this into context

Definition [Interest]:
The *interest* of a rule $I \rightarrow j$ is defined as

$$\frac{\text{supp}(I \cup \{j\})}{\text{supp}(I)} - \frac{\text{supp}(\{j\})}{n} \tag{3}$$

that is the confidence of $I \rightarrow j$ minus the fraction of baskets that contain $j$

# Association Rules: Interest

Definition [Interest]:
The *interest* of a rule $I \to j$ is defined as

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that is the confidence of $I \to j$ minus the fraction of baskets that contain $j$

*Examples*

- $\{\textit{diapers}\} \to \textit{beer}$ was found to have great interest
- $\{\textit{dog}\} \to \textit{cat}$ has interest $5/7 - 3/4 = -0.036$
- $\{\textit{cat}\} \to \textit{kitten}$ has interest $1/6 - 1/8 = 0.042$

# FREQUENT ITEMSETS TO ASSOCIATION RULES

*Situation*

- Consider frequent itemsets of "reasonably high" support $s$
  - Note that each frequent itemset suggests to be acted upon
    ☞ keep their number reasonably low
  - Reasonably high often means about 1% of baskets
- Confidence for a rule $I \to j$ should be at least (about) 50%
  ☞ Support for $I \cup \{j\}$ also fairly high

*Procedure*

- Assume all $I$ with $\text{supp}(I) \geq s$ have been mined
- For $J$ of $n$ items with $\text{supp}(J) \geq s$, there are $n$ possible association rules $J \setminus \{j\} \to J$
- $\text{supp}(J) \geq s$ implies $\text{supp}(J \setminus \{j\}) \geq s$
- Confidence of $J \setminus \{j\} \to J$ is easily computed as

$$\frac{\text{supp}(J)}{\text{supp}(J \setminus \{j\})}$$

*Mining Frequent Itemsets*
*The A-Priori Algorithm*

# Market-Basket Data: Representation

- Market-basket data is stored in a file basket-by-basket
  - If items refer to identifiers, for example $\{3, 36, 99\}\{6, 78, 11\}\ldots$
- *Assumption:* Average size of basket is rather small
- *Usually,* file does not fit in main memory
- Generating all subsets of size $k$ for a basket of size $n$ requires

$$\binom{n}{k} \approx \frac{n^k}{k!}$$

  runtime
- This often is little time because
  - $n$ was assumed to be small
  - $k$ is usually very small
  - When $k$ is large, one can virtually reduce $n$ further by removing infrequent items

# Market-Basket Data: Runtime Consideration

*Insight*

- Runtime is dominated by transferring data from disk to main memory
- *Consequence:* Processing all baskets is proportional to size of file
- *Runtime of algorithm* is proportional to number of passes through file
- For a *fast frequent itemset mining* algorithm:

**Limit number of passes through basket file**

# Use of Main Memory

- *Issue:* One needs to store counts for itemsets of size $k$
  - There could be many such itemsets
  - How to store these counts?
- *Consequence:* There is a limit on the number of items an algorithm can deal with
- *Example:*
  - Let there be $n$ items
  - For counting pairs, we need to store $\binom{n}{2} \approx n^2/2$ counts
  - Integers of 4 bytes: need $2n^2$ bytes to store counts
  - Consider machine of 2 GB, or $\approx 2^{31}$ bytes of main memory
  - Then $n < 2^{15} \approx 33\,000$ is required
- *Note:* Items can be hashed to integers, if they are not integers

# Storing Itemset Counts: The Triangular-Matrix Method

- In the following, consider storing itemsets of size 2
  - Remember that support threshold is quite large in real applications
  - So, many more pairs than triples, quadruples and so on in real applications
- *Insight:* Storing counts $a[i,j]$ in matrix $A = (a[i,j])_{1 \leq i < j \leq n} \in \mathbb{N}^{n \times n}$ wastes half of $A$
- *Solution:* Store count for pair of items $\{i,j\}, 1 \leq i < j \leq n$ in

$$a[k] \quad \text{where} \quad k = (i-1)(n - \frac{i}{2}) + j - i \tag{4}$$

This stores pairs in lexicographical order

$$\{1,2\}, \{1,3\}, ..., \{1,n\}, \{2,3\}, ..., \{2,n\}, ..., \{n-2,n\}, \{n-1,n\}$$

# Storing Itemset Counts: The Triples Method

- Store triples $[i, j, c]$ for all pairs $\{i, j\}$ whose count $c > 0$
- For example, do this with hash table, hashing $i, j$ as search key
- *Advantage:* Does not require space for pairs $\{i, j\}$ of count zero
- *Disadavantage:* Requires three times the space if $c > 0$
- *Rationale:* Triangular matrix method better if at least 1/3 of the $\binom{n}{2}$ pairs appear in basket

# Storing Itemset Counts: Example

*Example*

- Consider
  - 100 000 items
  - 10 000 000 baskets of
  - 10 items each
- Triangular-matrix method: $\binom{10^5}{2} \approx 5 \times 10^9$ integer counts
- Triples method: $10^7 \binom{10}{2} \approx 4.5 \times 10^8$ counts, making for $3 \times 4.5 \times 10^8 = 1.35 \times 10^9$ integers to be stored
- Triples method proves to be more appropriate

# MONOTONICITY

THEOREM [MONOTONICITY]:

- Let $s$ be the support threshold.
- Let $I, J$ be sets such that $J \subseteq I$

Then if $I$ is frequent, any subset $J$ of $I$ is, too:

$$\operatorname{supp}(I) \geq s \quad \text{implies} \quad \operatorname{supp}(J) \geq s \tag{5}$$

PROOF.
Each basket that holds $I$ also holds $J$, as $J$ is contained in $I$. So, the number of baskets that hold $J$ is at least as large as the number of baskets that hold $I$. □

# Maximal Frequent Itemset

Definition [Maximal Frequent Itemset]:

- Let $s$ be the support threshold.
- Let $I$ be frequent, that is $\text{supp}(I) \geq s$.

$I$ is said to be *maximal* if no superset of $I$ is frequent:

$$\text{for all } J \supsetneq I : \text{supp}(J) < s \quad (6)$$

*Example (from above):*

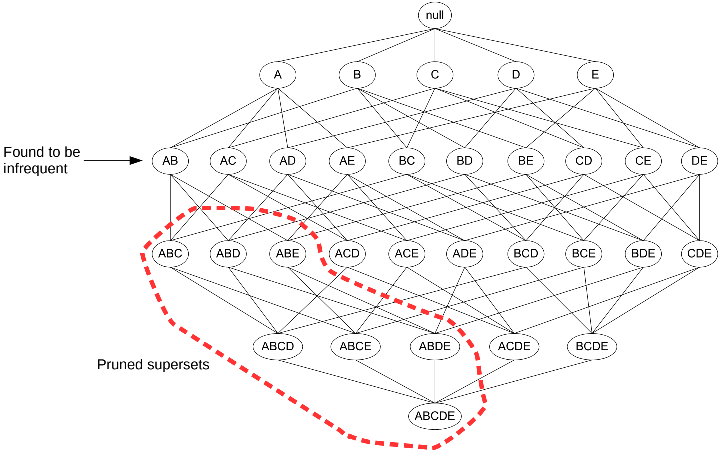
- At support threshold $s = 3$, we found frequent pairs $\{dog, a\}, \{dog, and\}, \{dog, cat\}, \{cat, a\}, \{cat, and\}$
- $\{dog, cat, a\}$ was found the only frequent triple

☞ $\{dog, cat, a\}, \{dog, and\}$ and $\{cat, and\}$ are maximal, while $\{dog, a\}, \{dog, cat\}, \{cat, a\}$ are not

# Note on Counting Pairs

- The number of frequent pairs is larger than frequent triples, quadruples, ... why?
- For making sense, number of (maximal) frequent itemsets is supposed to be sufficiently small
  - Human applicants need to work it out on all of them
- So, support threshold is set sufficiently high
- Any maximal frequent itemset holds many more smaller, non-maximal frequent itemsets
- The resulting situation implies that there are many more frequent pairs than triples, many more frequent triples than quadruples, and so on
- *Important:*
  - Still, the possible number of triples, quadruples is (much) greater than pairs
  - Any good frequent itemset *algorithm needs to avoid running through all possible triples, quadruples, and so on*

# Monotonicity to the Rescue




Itemsets for items A,B,C,D,E

Neglecting supersets of infrequent pair {A,B}

Adopted from `mmds.org`

# A-Priori Algorithm: Motivation

In the following, we focus on determining frequent pairs.

## *N*aive Approach

Consider the algorithm

- For each basket, use double loop to generate all pairs contained in it
- For each pair generated, add 1 to its count
- Store counts using triangular or triples method
- At the end, run through all pairs and determine those whose counts exceed support threshold $s$
- *Benefit:* Only one pass through all baskets
- *Issue:* Number of pairs considered usually does not fit in main memory

# A-Priori Algorithm: Motivation

In the following, we focus on determining frequent pairs.

*N*aive Approach

- *Possible Benefit:* Only pass through all baskets
- *Issue:* Number of pairs considered usually does not fit in main memory

*S*olution: A-Priori-Algorithm

- Have *two passes through baskets* instead of one
- In first run, determine candidate pairs, for which counts are stored
- In second run, determine counts for candidate pairs
- Finally filter for frequent pairs

# A-Priori Algorithm: First Pass

*Create and Maintain Two Tables*

- *First table A:* Let $x$ be an item name, then $A[x]$ reflects that $x$ is the $A[x]$-th item in the order of their appearance in the basket file
- *Second table B:* Let $k$ be an item number, then $B[k]$ is the number of baskets in which item number $k$ appears

*Read Baskets: Fill Table B*

- For each basket, for each item $x$ in the basket, do

$$B[A[x]] = B[A[x]] + 1 \quad (7)$$

- That is, iteratively increase item counts while running through all items in all baskets

# A-Priori Algorithm: Second Pass I

- Let $n$ be the number of items
- Let $m$ be the number of items found to be *frequent*
- By user constraints, usually $m << n$

*Create Third Table*

- *Third table C:* Let $1 \leq k \leq n$ be an item number. Then

$$C[k] = \begin{cases} 0 & \text{if item number } k \text{ is not frequent} \\ l & \text{if item number } k \text{ was found the } l\text{-th frequent item} \end{cases} \tag{8}$$

So, $C \in \{0, 1, ..., m\}^n$, where

  - $C[k] = 0$ $n - m$ times
  - $C[k] = i, 1 \leq i \leq m$ exactly one time
  - $0 < C[k_1] < C[k_2]$ implies $k_1 < k_2$, expressing that $C$ preserves the order of appearance of items

# A-PRIORI ALGORITHM: SECOND PASS II

*Count Pairs Data Structure*

- Use either triangular or triples method data structure to hold counts
  - For using triangular method, renumbering necessary
- By monotonicity, a pair can only be frequent, if both items are frequent
- So, space required is $O(m^2)$ rather than $O(n^2)$
  ☞ $m << n$ implies $m^2 << n^2$, so fits in main memory!

*Examine Baskets*

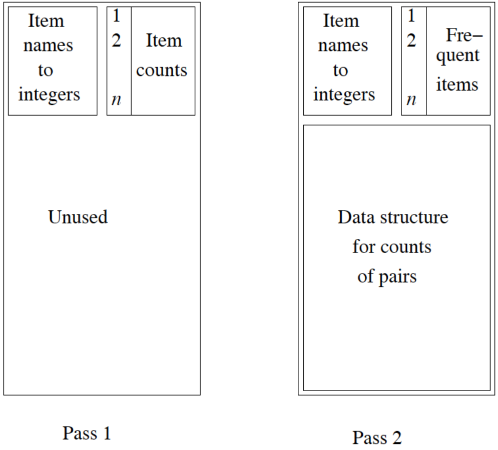
1. For each basket, for each item $x$, see whether

$$C[A[x]] > 0 \quad \text{that is, whether } x \text{ is frequent} \tag{9}$$

2. Using double loop, generate all pairs of frequent items in the basket
3. For each such pair, increase count by one in pair count data structure

*Eventually:* examine which pairs are frequent in pair count data structure

# A-Priori Algorithm: Main Memory Usage



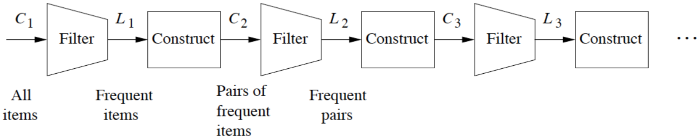

Use of main memory during A-Priori passes

Adopted from `mmds.org`

# A-PRIORI ALGORITHM: ALL FREQUENT ITEMSETS

- *One extra pass* for each $k > 2$ to mine frequent itemsets of size $k$
- The A-Priori algorithm proceeds iteratively
  - Mining frequent itemsets of size $k + 1$ is based on knowing frequent itemsets of size $k$
- Each iteration consists of two steps for each $k$:
  - Generate a candidate set $C_k$
  - Filter candidate set $C_k$ to produce $L_k$, the truly frequent itemsets of size $k$
- The algorithm terminates at first $k$ where $L_k$ is empty
  - Monotonicity says we are done mining frequent itemsets

# A-PRIORI ALGORITHM: CANDIDATE GENERATION AND FILTERING




A-Priori algorithm: Alternating between candidate generation and filtering

Adopted from `mmds.org`

- *Construct:* Let $C_k$ be all itemsets of size $k$, every $k-1$ of which belong to $L_{k-1}$
- *Filter:* Make a *pass through baskets* to count members of $C_k$; those with count exceeding $s$ will be part of $L_k$
  - For storing counts for itemsets of size $k$, extend triples method
  - E.g. storing quadruples for frequent triples, and so on...

*A-Priori Algorithm Extensions*
*The PCY Algorithm*

# Bottleneck: Size of $C_2$

- The predominant bottleneck in most applications of A-Priori is the size of $C_2$, the candidate pairs
- Several algorithms address to trim down that size
- Exemplary algorithms:
  - The algorithm of Park, Chen and Yu *(PCY algorithm)*
  - The Multistage algorithm
  - The Multihash algorithm
- We will briefly treat the PCY algorithm here

# The PCY Algorithm

- *Observation:* Much of main memory during first pass of A-Priori remains unused
- Use that space for a hash table $H$ that
  - hashes pairs of items $\{i, j\}$ to
  - buckets holding integers $H[\{i, j\}] \in \mathbb{N}$, where

$$H[\{i, j\}] \quad \text{is number of times any pair hashed to that bucket} \tag{10}$$

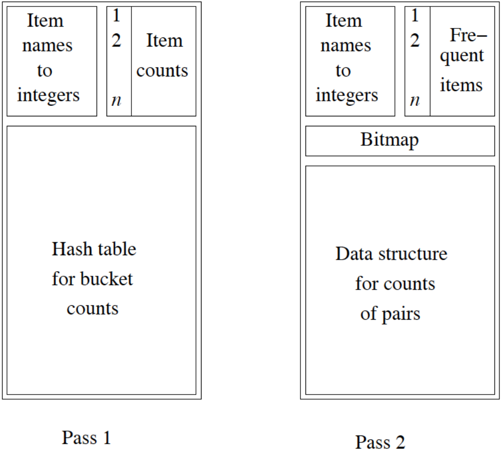
- To construct $H$, use double loop through baskets:
  - hash each resulting pair to bucket
  - increase the integer in that bucket by one
- A *frequent bucket $b$* exceeds the support threshold $s$

# The PCY Algorithm

- A *frequent bucket* $b$ exceeds the support threshold $s$
- So, for any bucket $b$:
  - If $b$ is infrequent, none of the pairs that hashed to $b$ are frequent
  - If $b$ is frequent, pairs hashing to it could be frequent
- *Definition of* $C_2$*:* For $\{i, j\} \in C_2$, both
  - $i$ and $j$ must be frequent
  - $\{i, j\}$ must hash to a frequent bucket
- *Use of* $C_2$ *in second pass:*
  - Transform $H$ into bitmap $H'$

$$H'[\{i,j\}] = \begin{cases} 1 & \text{if } H[\{i,j\}] \geq s \\ 0 & \text{if } H[\{i,j\}] < s \end{cases} \tag{11}$$

# PCY Algorithm: Main Memory Usage




Use of main memory during A-Priori passes

Adopted from `mmds.org`

# MATERIALS / OUTLOOK

- See *Mining of Massive Datasets*, sections 6.1, 6.2, 6.3.1, 6.4.1, 6.4.2, 6.4.5
- As usual, see `http://www.mmds.org/` in general for further resources
- Next lecture: 'Recommendation Systems"
  - See *Mining of Massive Datasets*, 9.1, 9.3, 9.4